Solution:

The decimal values for which the function f assumes value '1' are 0,1,4,5

$$= \Sigma\ (0,1,4,5)$$

$$\therefore \quad f = 000 + 001 + 100 + 101 = \text{sum of minterms}$$

$f(A,B,C) = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + A\bar{B}\bar{C} + A\bar{B}C$ which is the SOP form and can be minimized to

$$f = \bar{A}\bar{B}\,(\bar{C} + C) + A\bar{B}\,(\bar{C} + C)$$

$$= \bar{A}\bar{B} + A\bar{B} = \bar{B}(\bar{A} + A) = \bar{B}\,.\,1 = \bar{B}$$

The decimal values for which the function f assumes value '0' are 2,3,6,7 = II (2,3,6,7)

$$\therefore \quad f = II(2,3,6,7) \text{ is the product of maxterms form}$$

$$= (010)\,(011)\,(110)\,(111)$$

$f(A,B,C) = (A+\bar{B}+C)\,(A+\bar{B}+\bar{C})\,(\bar{A}+\bar{B}+C)\,(\bar{A}+\bar{B}+\bar{C})$ which is the POS form of the switching function and it can be minimized to

$$f(A,B,C) = [(A+\bar{B})\bar{C} + (A+\bar{B})C + C\bar{C}]\,[(\bar{A}+\bar{B})\bar{C} + (\bar{A}+\bar{B})C + C\bar{C}]$$

$$= [(A+\bar{B})\,(\bar{C}+C) + 0]\,[(\bar{A}+\bar{B})\,(\bar{C}+C) + 0]$$

$$= (A+\bar{B})\,.\,1\,.\,(\bar{A}+\bar{B})\,.1 = A\bar{A} + A\bar{B} + \bar{B}\bar{A} + \bar{B}\bar{B}$$

$$= 0 + \bar{B}(A+\bar{A}) + \bar{B} = \bar{B} + \bar{B} = \bar{B}$$

2.13 KARNAUGH MAP—CONSTRUCTION AND PROPERTIES

The Karnaugh map is a modified form of the Venn diagram of the switching function with four or less variables in the canonical sum of products form. When the Venn diagram of a switching function is redrawn using rectangles and squares, and complemented and uncomplemented variables are represented by 0's and 1's respectively, in the column or rows which it represents, the diagram is called a Karnaugh map. Each square of the Karnaugh map is denoted by a binary number or its equivalent decimal. If the switching function consists of three-variables, then the Karnaugh map has $8(=2^3)$ squares and if the function consists of four-variables, then the map has $16(=2^4)$ small squares. To construct the Karnaugh map of a switching function, first the function is represented in the sum of products (minterms) form. For example if the function is represented by:

$$f = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}B\bar{C} + \bar{A}BC + ABC$$

$$= 000 + 001 + 010 + 011 + 111$$

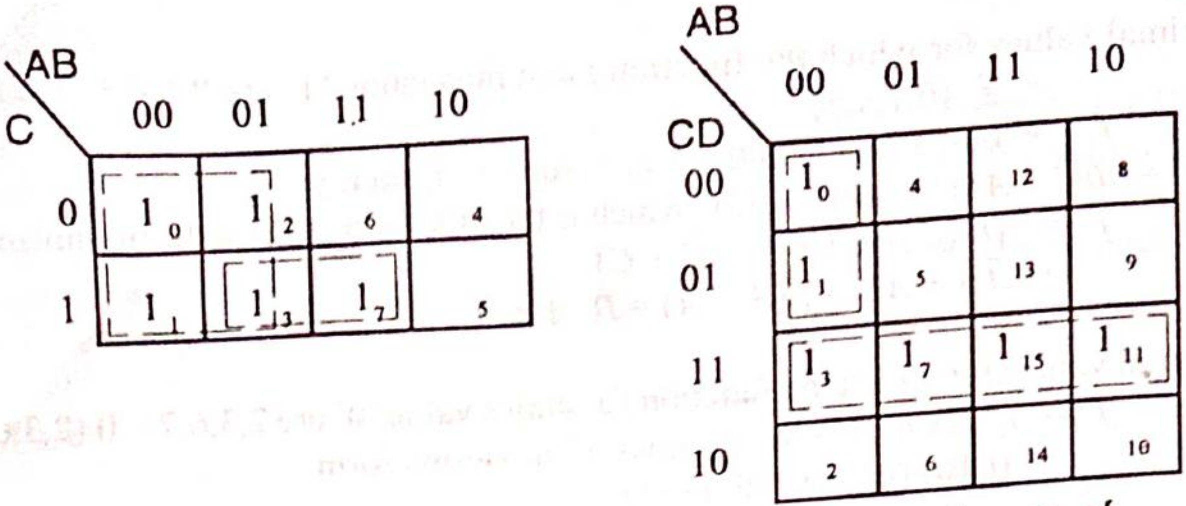
$$= \Sigma\ (0,1,2,3,7)$$

Then the above function $f(A,B,C)$ is represented on the Karnaugh map shown in Fig. 2.14(a) by marking those squares by '1', which represent each product term of the function in decimal form viz. 0,1,2,3,7. These decimal numbers are usually written in small numerals at the bottom right corners of the squares representing a particular decimal number. Similarly Fig. 2.14 (b) shows the Karnaugh map of a four-variable switching function represented by

$$f = \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}\bar{B}CD + \bar{A}BCD + A\bar{B}CD + ABCD$$

$$= 0000 + 0001 + 0011 + 0111 + 1011 + 1111$$

$$= \Sigma(\,0,1,3,7,11,15)$$



(a)Karnaugh map of $\Sigma\,(0,1,2,3,7)$

(a)Karnaugh map of $\Sigma\,(0,1,3,7,11,15)$

Fig. 2.44

The Karnaugh map of the above four-variable switching function [Fig. 2.44(b)] is constructed by filling 16(=2^4) squares.The switching function is represented on the map by marking those squares by '1' which represent the product terms of function in the decimal form viz., 0,1,3,7,15 which are written in small numerals at the bottom right corners of these squares. While constructing Karnaugh map it is customary to indicate only all the 1's in the squares that correspond to the product terms in the canonical sum form of the given switching function.

After constructing the Karnaugh map of a given switching function with four or less variables, we would like to use it in the application of minimization of the function. The following properties, depending on the relative positions of the squares marked by '1', when adjacent, lead to simplification of the switching function, by combining these '1' cells into pairs, quartets or octets as the case may be.

1. An adjacent pair of '1' cells in a Karnaugh map can be combined to form a 'pair', simplifying the two product terms into one term, thus eliminating one variable. For example the product terms 3 and 7 representing the terms $\bar{A}BC$ and ABC respectively are adjacent on the Karnaugh map in Fig. 2.44(a) and can thus be combined to form a pair of '1' cells. This is shown by enclosing these two '1' cells by a dotted rectangle around them. This pair corresponds to BC. Algebraically also using Boolean algebra we get $\bar{A}BC + ABC = (\bar{A}+A)\,BC = BC$. Thus two product terms $\bar{A}BC$ and ABC have been simplified into one term (= BC), eliminating one variable (A) here.

2. An adjacent pattern or combination of four '1' cells in a Karnaugh map can be combined to form a quartet, simplifying the sum of four product terms into one term only, thus eliminating two variables. For example the product terms 0,1,2,3 representing the terms $\bar{A}\bar{B}\bar{C}$, $\bar{A}\bar{B}C$, $\bar{A}B\bar{C}$ and $\bar{A}BC$ in order form an adjacent pattern of four '1' cells as shown on the Karnaugh map of Fig. 2.44(a). They can be combined to form a quartet of four '1' cells as shown in Fig. 2.44(a) by enclosing these four cells by a dotted square around them. This dotted square or quartet of '1' cells corrspond to A, because the variable A is necessarily '0' in all of these four places. Algebraically also, using Boolean algebra we have: $\bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}B\bar{C} + \bar{A}BC$.

$$= \bar{A}\bar{B}(\bar{C} + C) + \bar{A}B(\bar{C} + C)$$
$$= \bar{A}\bar{B} + \bar{A}B = \bar{A}(\bar{B} + B) = \bar{A}\cdot 1 = \bar{A}$$

3. An adjacent pattern or combination of eight '1' cells in a Karnaugh map can be combined to form a octet, simplifying the sum of eight product terms into one term only, thus eliminating three variables.

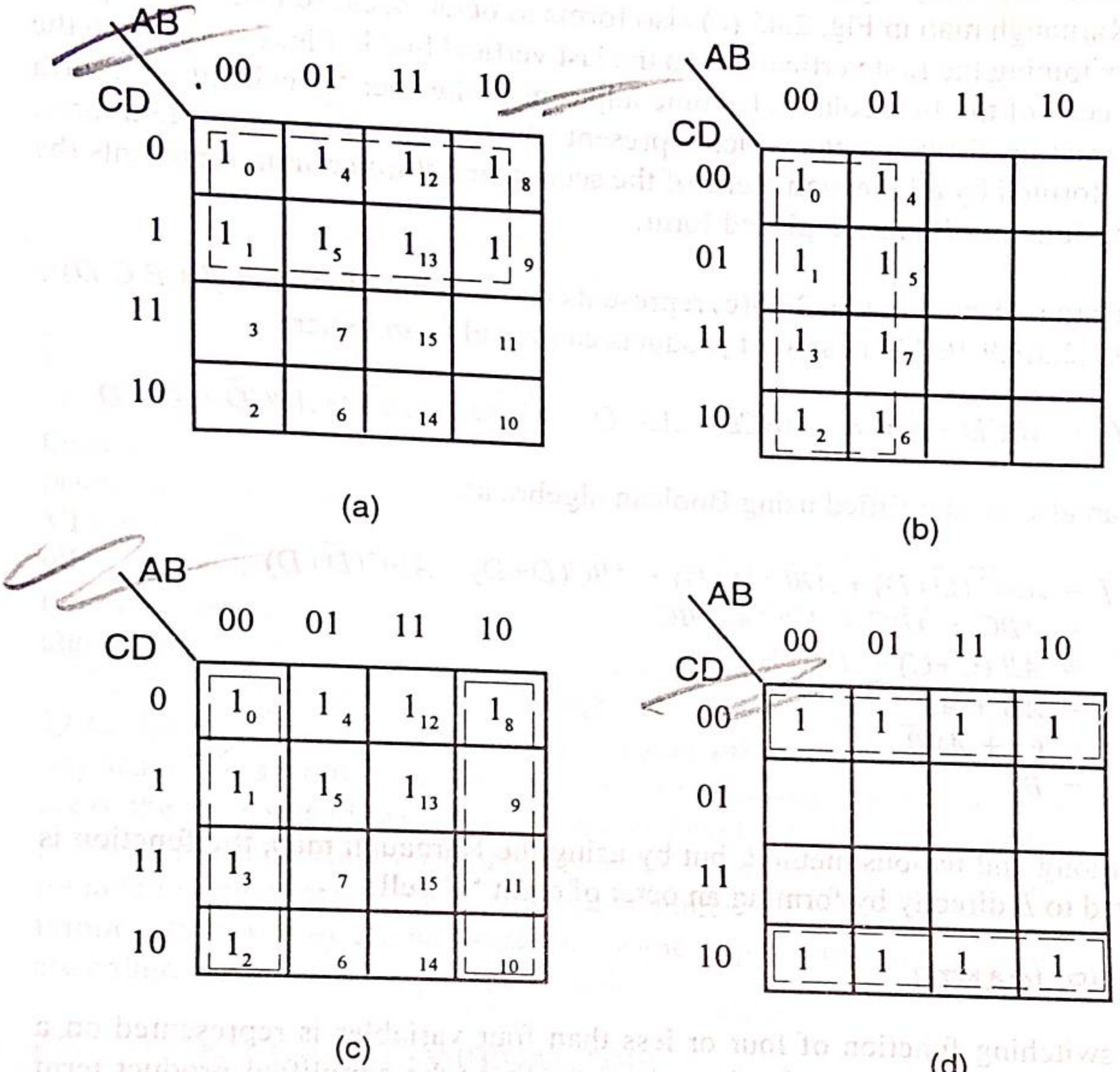


Fig. 2.45

An octet of eight '1' cells can be formed by any of the four patterns shown in Fig. 2.45. For example, the Karnaugh map in Fig. 2.45(a) represents 8 product terms: $\Sigma(0,1,4,5,8,9,12,13)$ representing the switching function $f(A,B,C,D)$, where

$$f = \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}B\bar{C}\bar{D} + \bar{A}B\bar{C}D + A\bar{B}\bar{C}\bar{D} + A\bar{B}\bar{C}D + AB\bar{C}\bar{D} + AB\bar{C}D$$

The Karnaugh map of this function shows that an octet can be formed, in which variable C is necessarily '0', thus representing $\bar{C}$.

If an octet is formed by filling all the eight squares of the second and third row by marking '1', then the Karnaugh map will represent the simplified or minimized function D. An octet formed by all the eight cells of third and fourth row will represent C. Similarly an octet formed by all the eight cells of first and fourth row will represent $\bar{D}$ as is shown in Fig. 2.45(d). These eight '1' cells form an octet, because if the paper is folded by joining the top line of first row with the bottom line of fourth row, then the four '1' cells of the first row became adjacent to four '1' cells in the fourth row.

An octet of eight '1' cells can also be formed column wise as shown in Fig. 2.45(b) and 2.45(c). The Karnaugh map in Fig. 2.45 (b) represents the function Σ (0,1,2,3,4,5,6,7) in SOP form and can be minimized to $\overline{A}$.

An octet formed by eight cells of the third and fourth column will also represent A. The Karnaugh map in Fig. 2.45 (c) also forms as octet, because when the paper is folded by joining the first vertical line to the last vertical line in Fig. 2.45(c), then the four '1' cells of the first column become adjacent to the four '1' cells of the fourth column and they form an octet which represents the function $\overline{B}$ in a simplified form. An octet formed by all the eight cells of the second and third column represents the switching function B in a simplified form.

The Karnaugh map in Fig. 2.45(c) represents the switching function $f(A,B,C,D) = \overline{B} = \Sigma(0,1,2,3,8,9,10,11)$ in sum of products canonical form, where

$$f = \overline{A}\overline{B}\overline{C}\overline{D} + \overline{A}\overline{B}\overline{C}D + \overline{A}\overline{B}C\overline{D} + \overline{A}\overline{B}CD + A\overline{B}\overline{C}\overline{D} + A\overline{B}\overline{C}D + A\overline{B}C\overline{D} + A\overline{B}CD$$

which can also be simplified using Boolean algebra as:

$$\begin{aligned} f &= \overline{A}\overline{B}\overline{C}(\overline{D}+D) + \overline{A}\overline{B}C(\overline{D}+D) + A\overline{B}\overline{C}(\overline{D}+D) + A\overline{B}C(\overline{D}+D) \\ &= \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}C + A\overline{B}\overline{C} + A\overline{B}C \\ &= \overline{A}\overline{B}(\overline{C}+C) + A\overline{B}(\overline{C}+C) \\ &= \overline{A}\overline{B} + A\overline{B} \\ &= (\overline{A} + A)\overline{B} \\ &= \overline{B} \end{aligned}$$

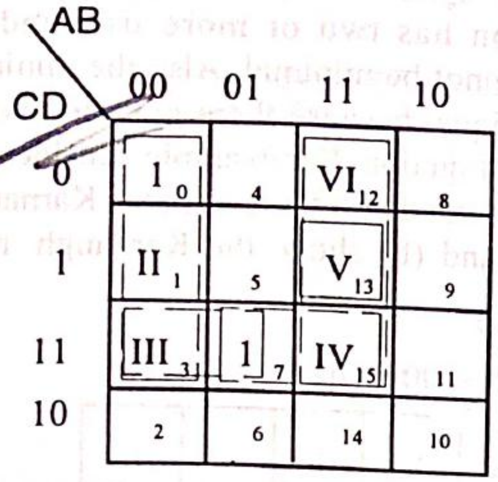
This is a long and tedious method, but by using the Karnaugh map, the function is minimized to $\overline{B}$ directly by forming an octet of eight '1' cells.

2.14 IMPLICANTS

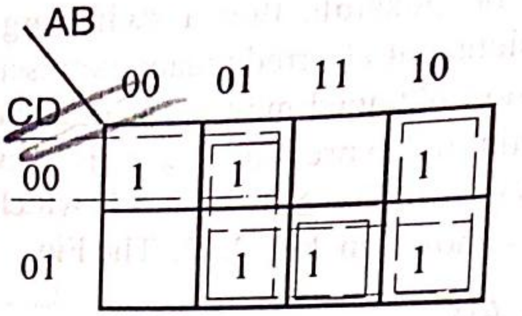
When a switching function of four or less than four variables is represented on a Karnaugh map, then the set of adjacent minterms or the simplified product term obtained by combining the minterms of set are called implicants of the switching function.

Prime-implicant An implicant is called a prime-implicant of the switching function if it is not a subset of any other implicant of the switching function. Each product term individually is a prime-implicant. From the prime-implicant of a switching function, deletion of any variable or literal is not possible.

Essential prime-implicant A prime-implicant which includes a '1' cell, which is not included in any other prime-implicant, on the Karnaugh map, is known as an essential prime-implicant of the switching function.



(a) Karnaugh map of
Σ(0,1,3,7,12,13,15)



(b) Karnaugh map of
Σ(0,2,3,4,5,7)

Fig. 2.46

For example, the Karnaugh map of $f(A,B,C,D) = \Sigma\ (0,1,3,7,12,13,15)$ is shown in Fig. 2.46(a). It has six prime-implicants I, II, III, IV, V and VI each formed by a pair of two adjacent '1' cells i.e., by combining a set of two adjacent minterms. Out of these six prime-implicants only I and VI are essential prime-implicants. Because prime-implicant I includes a '1' cell (corresponding to 0 = 0000) and prime implicant VI includes a '1' cell (corresponding to 12 = 1100), which are not included in any other prime- implicant, therefore I and VI are essential prime-implicants. The essential prime-implicant must contain at least one minterm (a '1' cell) which is not included in any other prime-implicant of the switching function.

Figure 2.46(b) shows the Karnaugh map of the switching function $f(A,B,C,) = \Sigma(0,2,3,4,5,7)$. This map is known as the cyclic prime-implicant map. Here no prime-implicant is essential. The function has six prime implicants. All prime implicants are of the same size of two '1' cells each and every '1' cell is covered by exactly two prime implicants. This Karnaugh map shows that there is no minterm, which is not included in other prime implicants. Each minterm is included in at least two prime implicants, thus they are non-essential prime implicants and such prime-implicants are called cyclic prime implicants.

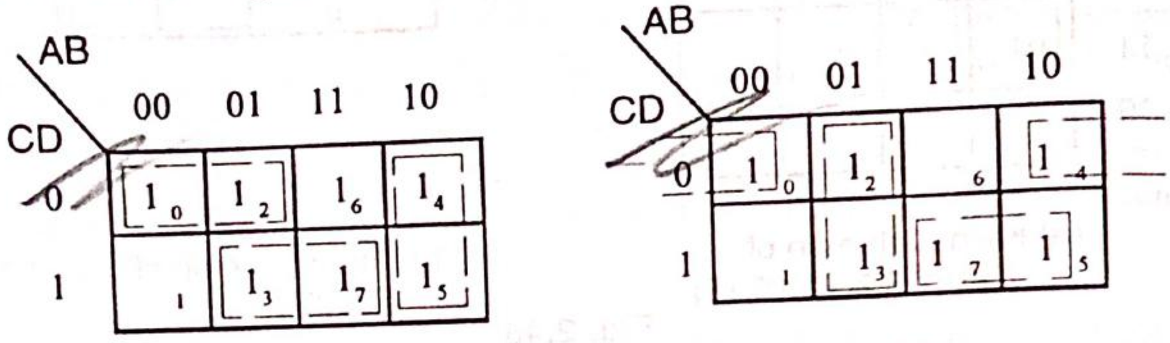
2.15 DON'T CARE COMBINAITONS

Sometimes, a situation occurs, when a function assumes the value '1' for some combinations and value '0' for other combinations. Thus the function can assume either a '0' or '1' value for a number of combinations; under the situation when the variables are not mutually independent. The combinations for which the value of the function is not specified with certainity are called don't care combinations. There values are denoted by ϕ or D on the Karnaugh map. We may assign value '1' to some selected don't care combination to make bigger sub-cubes. A sub-cube containing only don't care cells cannot be formed.

2.16 IRREDUNDANT EXPRESSION

A sum of product of expression from which no variable or literal can be deleted without changing its logical value, is called an irredundant or irreducible expression.

It may be possible that a switching function has two or more irredundant expressions but all irredundant expressions may not be minimal. Also the minimal expression obtained may not necessarily be unique, because there may be two or more minimal expressions for a given switching function. For example consider the function $f(A,B,C) = \Sigma\,(0,2,3,4,5,7)$ which is represented on a three-variable Karnaugh map as shown in Fig. 2.47. The Figs. 2.47(a) and (b) show the Karnaugh map



(a) (b)

Fig. 2.47

forming different sets of sub-cubes of two cells for the same switching function $f(A,B,C)$ where

$$f(A,B,C) = \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC + A\bar{B}\bar{C} + A\bar{B}C + ABC$$

The Karnaugh map in Fig. 2.47(a) gives the irredundant minimal expression for $f(A,B,C) = \bar{A}\bar{C} + A\bar{B} + BC$ and the Karnaugh map in Fig. 2.47(b) gives another irredundant minimal expression for the function as $f(A,B,C) = \bar{B}\bar{C} + \bar{A}B + AC$.

Thus it is seen that the two irredudant expressions obtained for the switching function are both minimal, but they are not a unique minimal expressions. Therefore the minimal expression obtained may not be necessarily unique.

2.17 MINIMIZATION IN SOP FORM USING KARNAUGH MAP

Minimization or simplification of Boolean expressions and switching functions has been done using postulates of Boolean algebra and De Morgan's Theorems. This algebraic method of minimization is simple to work at, when the number of variables in a function are two or at the most three. As the number of variables in a function increases, the algebraic methods become tedious and more time consuming.

The Karnaugh map method of minimization of functions and expressions is a very simple and useful method if the number of variables are limited to four. Any function to be minimized or to be reduced is first written in sum of product form and its minterms are written. Each minterm is then converted to its equivalent binary number and a marking '1' is made in the corresponding square in the map, or by obtaining the corresponding coincidence of the variables in each term. The drawing of the Karnaugh map and its use to minimize will be clear from the following examples:

EXAMPLE **40**: Minimize the Boolean function
$f(A,B,C) = \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC$ using Karnaugh map

C \ AB	00	01	11	10
0	1 (0)	1 (2)	(6)	(4)
1	(1)	1 (3)	(7)	(5)

Fig. 2.48 Karnaugh map of function F

Solution:

The binary numbers along the top of the map indicate the conditions of variables *A* and *B* for each column and the binary numbers, 0 and 1 written alongside vertically indicate the condition of variable *C* along each of the two rows.

Now $f(A,B,C) = \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC$
$= 000 + 010 + 011$
$= \Sigma\,(0,2,3)$

The term $\bar{A}\bar{B}\bar{C}$ has the corresponding binary number 000 i.e., A, B and C values are all '0'. Therefore a marking 1 is marked in the first square in the first column from left (indicating $AB = 00$) and in the first row from the top (indicating C value $= 0$). The second way of recognising the square is by marking '1' in the square which corresponds to the decimal value of the term $\bar{A}\bar{B}\bar{C} = 000 = 0$. To illustrate this, decimal values corresponding to each square are first written at the right bottom corner of each square. Similarly the term $\bar{A}B\bar{C} = 010 = 2$ in decimal value, therefore the marking '1' is marked in that square, in which 2 is written in the right bottom corner. The term $\bar{A}BC = 011 = 3$ in decimal value, therefore marking '1' is marked in the square in which decimal 3 is written in the right bottom corner. Thus all the three minterms of the function have been shown on the Karnaugh map by marking '1' in the square corresponding to these three terms.

Two cells are said to be adjacent if they differ in just one variable. As shown in Fig. 2.48 two terms $\bar{A}\bar{B}\bar{C}$ (=0) and $\bar{A}B\bar{C}$ (=2), differ in just one variable, are adjacent cells. These two terms occupy two adjacent cells in the first row as shown by '1' marked in them. Thus a pair of two '1' cells is made by marking a dotted enclosure around them. In this pair, variable A is necessarily 0 and variable C is also necessarily 0, therefore this pair corresponds to $\bar{A}\bar{C}$.

Similarly a vertical pair is formed in the second column by combining two adjacent '1' cells, shown by marking dotted enclosure around these '1' cells. In this pair, A is necessarily 0 and B is necessarily 1 as written above this column, but nothing can be said about C value with certainty. Thus this pair corresponds to $A = 0$ and $B = 1$, which means $\bar{A}B$.

Thus the Boolean function

$\bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC$ is minimized to $\bar{A}\bar{C} + \bar{A}B$

EXAMPLE 41: Minimize $F = \bar{A}BC + \bar{A}B\bar{C} + A\bar{B}\bar{C} + A\bar{B}C$ using the Karnaugh map or Reduce Σ (2,3,4,5) using the Karnaugh map.

Solution:

Function $f(A,B,C) = \bar{A}BC + \bar{A}B\bar{C} + A\bar{B}\bar{C} + A\bar{B}C$

$= 011 + 010 + 100 + 101$
$= \Sigma\,(3,2,4,5) = \Sigma\,(2,3,4,5)$

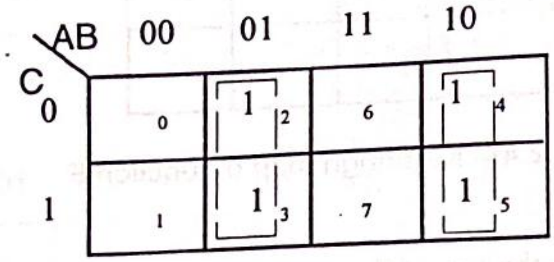


Fig. 2.49 Karnaugh map for (2,3,4,5)

To plot the Karnaugh map for the given function, marking '1' is marked in all those squares in which 2,3,4,5 decimal values are written in small numerals at the bottom right corner of the squares.

One pair is formed by combining two adjacent cells corresponding to the decimal values 2 and 3. This pair is shown by marking a dotted enclosure around these two '1' cells. This pair corresponds to $AB = 01$ written at the top of this column. Therefore the term corresponding to this pair is $\bar{A}B$.

The second pair is formed by combining two adjacent cells corresponding to the decimal values 4 and 5 in the last column. This pair is shown by marking a dotted enclosure around these two '1' cells. This pair corresponds to $AB = 10$ written at the top of this column. Therefore, the term corresponding to this pair is $A\bar{B}$.

Thus the Boolean function given by:

$$f = \bar{A}BC + \bar{A}B\bar{C} + A\bar{B}\bar{C} + A\bar{B}C = \Sigma\,(2,3,4,5) \text{ is minimized to } \bar{A}B + A\bar{B}.$$

EXAMPLE 42: Minimize the Boolean function.

$f = \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + A\bar{B}\bar{C} + AB\bar{C} = \Sigma\,(0,2,4,6)$ using the Karnaugh map

Solution:

The given function $f = \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + A\bar{B}\bar{C} + AB\bar{C}$
$= 000 + 010 + 100 + 110$
$= \Sigma\,(0,2,4,6)$

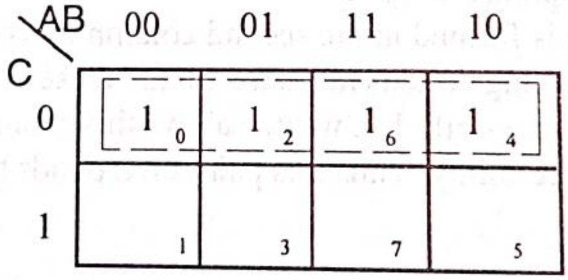


Fig. 2.50 Karnaugh map for Σ (0,2,4,6)

The Karnaugh map for $\Sigma(0,2,4,6)$ is plotted by marking '1' in those 4 squares in which decimal 0,2,4,6 are written in small numerals at the bottom right corners.

These four '1' cells are combined to form a quartet, which is shown by enclosing a dotted rectangle around these cells. This quartet occupies the whole of the first row, which corresponds to $C = 0$ i.e., $\bar{C}$.

Thus the function is minimized to $f = \bar{C}$.

EXAMPLE 43: Finding out the set of prime-implicants from the switching function given below, find out the minimal expression using the Karnaugh map.

$$f(A,B,C,D) = \Sigma\,(0,1,2,3,4,6,8,9,10,11)$$

Solution:
First the Karnaugh map of the above four-variable switching function is drawn as shown in Fig. 2.51.

CD \ AB	00	01	11	10
00	1_0	1_4		1_8
01	1_1			1_9
11	1_3			1_{11}
10	1_2	1_6		1_{10}

Fig. 2.51 Karnaugh map for Σ (0,1,2,3,4,6,8,9,10,11)

In the first step, the adjacent minterms are combined to form the following sub-cubes of two '1' cells to form the pairs.

(0,1), (0,4), (0,2), (0,8),(1,3), (1,9), (2,3), (2,6), (2,10), (3,11), (4,6), (8,9), (8,10), (9,11), (10,11).

In the second step, the adjacent sub-cubes of two cells obtained above in first step are combined to form the following sub-cubes of four cells to yield quartets of '1' cells.

(0,1,2,3), (0,1,8,9), (1,3,9,11), (0,2,4,6), (0,2,8,10), (2,3,10,11), (8,9,10,11).

The sub-cubes of two cells (pairs), which become included, while forming the sub-cubes of four cells, thus become redundant and therefore stand deleted from the set of pairs obtained in the first step. Here on examining, we find that all the sub-cubes of two cells (pairs) obtained in the first step are included in the sub-cubes of four cells. Thus all the pairs become redundant and stand deleted.

In the third step, the adjacent sub-cubes of four cells are combined, if possible to form the sub-cube of 8 cells (octet), given below. (0,1,2,3,8,9,10,11)

The sub-cubes of four cells (quartets or quads), which become included, while forming the sub-cubes of 8 cells, thus become redundant and therefore stand deleted from the set of quartets or quads obtained in the second step. Here on examining, we find that all the sub-cubes of four cells except (0,2,4,6) obtained in the second step are included in the sub-cube of 8 cells. Thus all quartets, except (0,2,4,6) become redundant and stand deleted.

Since, no bigger sub-cubes of 16 cells can be obtained, therefore, the sub-cubes of eight cells, sub-cubes of four cells and the sub-cubes of two cells, which are not deleted